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EXISTENCE AND NON-EXISTENCE OF GLOBAL SOLUTIONS FOR A SEMILINEAR HEAT EQUATION

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ABSTRACT

The existence and non-existence of global solutions and the L^p blow-up of non-global solutions to the initial value problem $u'(t) = \Delta u(t) + u(t)^\gamma$ on R^n are studied. We consider only $\gamma > 1$. In the case $n(\gamma - 1)/2 = 1$, we present a simple proof that there are no non-trivial global non-negative solutions. If $n(\gamma - 1)/2 \leqq 1$, we show under mild technical restrictions that non-negative L^p solutions always blow-up in L^p norm in finite time. In the case $n(\gamma - 1)/2 > 1$, we give new sufficient conditions on the initial data which guarantee the existence of global solutions.

1. Introduction

In this paper we study whether or not there exist solutions to the initial value problem

$$(1.1) \qquad u'(t) = \Delta u(t) + u(t)^\gamma, \qquad u(0) = \phi,$$

which are global in time, i.e. exist for all $t \geqq 0$. The solution is to be a curve in $L^p(R^n)$, some $p \geqq 1$, which assumes only non-negative values. We take γ to be bigger than 1.

Fujita, [2] and [3], has studied this question for classical solutions to (1.1). His results are as follows: If $n(\gamma - 1)/2 < 1$, then no non-negative global solution exists for any non-trivial initial data. If $n(\gamma - 1)/2 > 1$, then global solutions do exist for any non-negative initial data dominated by a sufficiently small Gaussian. The case $n(\gamma - 1)/2 = 1$ was decided by Hayakawa [5] for $n = 1, 2$ and by Kobayashi, Sirao and Tanaka [6] for general n. The result is that if $n(\gamma - 1)/2 = 1$, then no non-negative global solution exists for any non-trivial initial data. The

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proof of this result is very complicated, and even the more recent proof given by Aronson and Weinberger in the appendix to [1] is quite technical.

In this paper we present a considerably simplified proof for the case $n(\gamma-1)/2=1$. Also, under mild technical restrictions, we show that if $n(\gamma-1)/2 \leqq 1$, the non-negative solutions to (1.1) must blow-up in L^p norm in finite time. Finally, we give some new sufficient conditions for global existence in the case $n(\gamma-1)/2>1$.

2. Statement of results

We study (1.1) via the corresponding integral equation

$$u(t)=e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}(u(s)^\gamma)ds. \tag{2.1}$$

Recall that

$$e^{t\Delta}\phi(x)=\int_{R^n} G_t(x-y)\phi(y)dy, \qquad \text{where } G_t(x)=(4\pi t)^{-n/2}e^{-|x|^2/4t}.$$

For future reference we collect some well known facts about the semigroup $e^{t\Delta}$.

PROPOSITION 1. (a) $\|G_t\|_1=1$ *for all* $t>0$.

(b) *If* $\phi\geqq 0$, *then* $e^{t\Delta}\phi\geqq 0$ *and* $\|e^{t\Delta}\phi\|_1=\|\phi\|_1$.

(c) *If* $1\leqq p\leqq\infty$, *then* $\|e^{t\Delta}\phi\|_p\leqq\|\phi\|_p$ *for all* $t>0$.

(d) *If* $1\leqq p<q\leqq\infty$ *and* $r^{-1}=p^{-1}-q^{-1}$, *then* $\|e^{t\Delta}\phi\|_q\leqq(4\pi t)^{-n/2r}\|\phi\|_p$ *for all* $t>0$.

PROOF. Statement (a) is just the standard Gaussian integral. Statement (b) follows from Fubini's theorem and part (a). Statement (c) is just $\|G_t*\phi\|_p\leqq\|G_t\|_1\|\phi\|_p=\|\phi\|_p$. Finally, (d) follows by interpolating between the case $p=q$, which is part (c), and the case $p=1$, $q=\infty$, which is immediate.

Usually a solution to (2.1) in $L^p(R^n)$ will be a continuous curve $u:[0,T)\to L^p(R^n)$. However, we do not wish to exclude the possibility that it be only strongly measurable, in which case the integral equation might just hold for almost all t in the interval. There is some question as to how to interpret the integral expression. Ideally, we would like it to be a Bochner integral in L^p. Indeed, in many situations where solutions to this and similar integral equations are shown to exist (e.g. [8] and [9]), the integral turns out to be a Bochner integral. However, in deriving necessary conditions on ϕ for a solution to (2.1) to exist, we do not want to make such a restrictive interpretation of the integral.

For example, if $p \geqq \gamma$ and $u:[0,T)\to L^p$ is continuous, then $u(s)^\gamma \in L^{p/\gamma}$ and, because of the smoothing properties of $e^{t\Delta}$ (Proposition 1 (d)), the integrand is a continuous curve into L^p on $[0, t-\varepsilon]$ for $\varepsilon > 0$ and $t < T$. Thus, we might have to interpret the integral as an "improper" Bochner integral, i.e.

$$L^p - \lim_{\varepsilon \downarrow 0} \int_0^{t-\varepsilon} e^{(t-s)\Delta}(u(s)^\gamma)\, ds.$$

More generally, we might wish to interpret the integral as a weak integral or as a Bochner integral in L^q with $q \neq p$. As will be seen, the proof of the following theorem allows any reasonable interpretation of the integral.

THEOREM 1. *Suppose $n(\gamma-1)/2 = 1$ and that $\phi \geqq 0$ in $L^p(R^n)$ is not identically zero. Then there is no non-negative global solution $u:[0,\infty)\to L^p$ to the integral equation* (2.1) *with initial value ϕ.*

We remark that Fujita's result for $n(\gamma-1)/2 < 1$ has been shown to hold in the L^p setting. (See the remarks after corollary 5.1 in [9].) Also, these results on non-existence of global solutions depend on the fact that the underlying space is R^n. If R^n is replaced by a bounded domain with smooth boundary, then global solutions abound. (See [8], theorem 4.) The proof of Theorem 1 below is a simplification of my original proof which was suggested to me by T. Kato. I would like to thank Professor Kato for his remarks. As mentioned earlier, the first proofs of Theorem 1, in the classical setting, appeared in [5] and [6].

Let us turn now to L^p blow-up of solutions of the integral equation (2.1). For the moment we do not require that the solutions be non-negative, and so we replace the non-linear term $u(s)^\gamma$ by $|u(s)|^{\gamma-1}u(s)$. Theorems 1 and 4 in [8] and theorems 2 and 3 in [9] tell us that if $p > n(\gamma-1)/2$ and $1 \leqq p < \infty$, then for every $\phi \in L^p(R^n)$ there is a maximal continuous solution to (2.1) with initial value ϕ, $u:[0, T_\phi)\to L^p(R^n)$. If ϕ is non-negative, then so is $u(t)$. T_ϕ is the existence time of the trajectory; and if $T_\phi < \infty$, the theorems say that $\|u(t)\|_p \to \infty$ as $t \to T_\phi$. Furthermore, if $p \geqq \gamma$ then the solution curve is unique in the class of strongly measurable and locally essentially bounded curves $v:[0,T_\phi)\to L^p(R^n)$; and if $1 \leqq p < \gamma$ it is unique in the class of strongly measurable curves $v:(0,T_\phi)\to L^{p\gamma}(R^n)$ such that $t^{n(\gamma-1)/2p\gamma}\|v(t)\|_{p\gamma}$ is locally essentially bounded, including near 0. (Theorem 4 in [8] and theorem 3 in [9] are stated in the context of a bounded domain in R^n, but are easily adapted to $L^p(R^n)$. Also, in theorem 4 of [8] $p > \gamma$ is required, while we wish to allow $p \geqq \gamma$. This can be done since we are using the semigroup $e^{t\Delta}$, and may therefore substitute Proposition 1 (d) above for lemma 4.1 in [8]. Indeed, in the proof of Theorem 2 below we will need to sketch part of the proof of theorems 1 and 4 of [8] in the case $p = \gamma$; and this point will be made clear.)

If $n(\gamma-1)/2 < 1$, then $p > n(\gamma-1)/2$ for all $p \in [1,\infty)$. Since there are no non-trivial global non-negative solutions to (2.1) in this case, any non-negative solution in L^p meeting the technical requirements described above must blow-up in L^p norm in finite time. The same holds if $n(\gamma-1)/2 = 1 < p < \infty$. The only case not covered by the existing theorems is $p = n(\gamma-1)/2 = 1$. Indeed, it is not even clear in this case that for a given $\phi \in L^1(R^n)$ there exists a local solution to (2.1) with initial value ϕ. However, if $\phi \in L^1 \cap L^q$ for some $q > 1$, there is certainly a local solution to (2.1) with initial value ϕ; and by theorem 4 in [9], the solution remains in L^1 at least for some time. The following theorem gives a condition which guarantees that L^1 solutions blow-up in L^1 norm in finite time. As will be seen from the proof, the technical requirement in the hypothesis of the theorem is fairly natural.

THEOREM 2. *Suppose $n(\gamma-1)/2 = 1$ and that $u:[0,T) \to L^1(R^n)$ is a strongly measurable non-negative solution of the integral equation* (2.1) *with non-negative, non-trivial initial value $\phi \in L^1(R^n)$. It follows that $u(t) \in L^\gamma(R^n)$ for almost all $t \in (0,T)$. Suppose further that $u:(0,T) \to L^\gamma(R^n)$ is locally essentially bounded. Then $u(t)$ can be continued to a maximal solution of* (2.1) *in L^γ. This solution remains in L^1 throughout its trajectory and $\|u(t)\|_1 \to \infty$ as $t \to T_\phi$, where T_ϕ is the existence time of the trajectory.*

We also prove the following theorem, giving sufficient conditions for global solutions of (2.1) to exist.

THEOREM 3. (a) *Let $\phi \geqq 0$ be in $L^p(R^n)$, $1 \leqq p < \infty$. Suppose $(\gamma-1)\int_0^\infty \|e^{s\Delta}\phi\|_\infty^{\gamma-1} ds \leqq 1$. Then there exists a non-negative continuous curve $u:[0,\infty) \to L^p$ which is a global solution to* (2.1) *with initial value ϕ. Furthermore*

$$u(t) \leqq \frac{e^{t\Delta}\phi}{\left[1-(\gamma-1)\int_0^t \|e^{s\Delta}\phi\|_\infty^{\gamma-1} ds\right]^{1/(\gamma-1)}} \tag{2.2}$$

for all $t \geqq 0$.

(b) *Suppose $n(\gamma-1)/2 > 1$. If $\phi \geqq 0$ and $\|\phi\|_{n(\gamma-1)/2}$ is sufficiently small, then there exists a non-negative continuous curve $u:[0,\infty) \to L^{n(\gamma-1)/2}$ which is a global solution to* (2.1) *with initial value ϕ.*

It is a pleasure to thank Professor A. Browder for a helpful conversation concerning this work.

3. Proof of Theorem 1

We begin by recalling the essential elements of Fujita's proof in the case $n(\gamma-1)/2<1$. The crucial estimate is that if $u(t)$ is a non-negative solution to (2.1) on $[0, T)$, then

$$t^{1/(\gamma-1)}e^{t\Delta}\phi \leqq C \tag{3.1}$$

for all $t \in [0, T)$, where C is a fixed constant depending on γ but independent of ϕ and T. (See [3] p. 108 or [9] theorem 5.) Furthermore, if $\phi \geqq 0$ we clearly have

$$\lim_{t\to\infty}(4\pi t)^{n/2}e^{t\Delta}\phi = \|\phi\|_1 \tag{3.2}$$

pointwise on R^n. (The limit is of course infinite if ϕ is not in L^1.) If $n(\gamma-1)/2<1$, then (3.1) cannot possibly hold for large values of t because of (3.2). Thus, no non-negative global solutions exists.

Now suppose $n(\gamma-1)/2=1$ and that $u:[0,\infty)\to L^p$ is a global non-negative solution of (2.1) with initial value ϕ. Then (3.1) becomes

$$t^{n/2}e^{t\Delta}\phi \leqq C \tag{3.3}$$

for all $t \geqq 0$. Combining this with (3.2), we see that $\|\phi\|_1 \leqq C'$ for some fixed constant C'. Since $u(t)$ for (almost) any t can be regarded as the initial value, we must have

$$\|u(t)\|_1 \leqq C' \tag{3.4}$$

for (almost) all $t \geqq 0$.

Next, assume that the initial value ϕ dominates some Gaussian function, i.e. $\phi \geqq kG_\alpha$ for some $k>0$ and $\alpha>0$. (We will later see that this assumption is unnecessary for the final result.) From the integral equation (2.1) it follows that

$$u(t) \geqq e^{t\Delta}\phi \geqq e^{t\Delta}kG_\alpha,$$

and so

$$\begin{aligned}\|u(t)\|_1 &\geqq \int_0^t \|e^{(t-s)\Delta}u(s)^\gamma\|_1\,ds\\ &\geqq \int_0^t \|e^{(t-s)\Delta}(e^{s\Delta}kG_\alpha)^\gamma\|_1\,ds\\ &= k^\gamma\int_0^t \|(e^{s\Delta}G_\alpha)^\gamma\|_1\,ds.\end{aligned}$$

By the definition of the Gaussians G_t and their composition property under convolution, we get that

$$\begin{aligned}(e^{s\Delta}G_\alpha)^\gamma &= (G_{s+\alpha})^\gamma \\ &= [4\pi(s+\alpha)]^{-n(\gamma-1)/2}\gamma^{-n/2}G_{(s+\alpha)/\gamma} \\ &= [4\pi(s+\alpha)]^{-1}\gamma^{-n/2}G_{(s+\alpha)/\gamma}.\end{aligned}$$

Therefore, for almost all t,

$$\begin{aligned}\|u(t)\|_1 &\geqq k^\gamma\gamma^{-n/2}(4\pi)^{-1}\int_0^t (s+\alpha)^{-1}\|G_{(s+\alpha)/\gamma}\|_1\,ds \\ &= k^\gamma\gamma^{-n/2}(4\pi)^{-1}\int_0^t (s+\alpha)^{-1}\,ds,\end{aligned}$$

which gets arbitrarily large as $t\to\infty$. This contradicts (3.4) and proves Theorem 1 in the case where the initial data ϕ dominates some Gaussian.

To prove the result in general, given a non-negative solution $u(t)$ to (2.1) with non-trivial initial value ϕ, we consider $v(t)=u(t+\varepsilon)$ for some $\varepsilon>0$. Then $v(t)$ is a solution to (2.1) with initial value $\psi=u(\varepsilon)$. It suffices to show that $v(t)$ can not be a global solution, i.e. that ψ dominates some Gaussian function. But $\psi=u(\varepsilon)\geqq e^{\varepsilon\Delta}\phi=G_\varepsilon*\phi$, and

$$\begin{aligned}(G_\varepsilon*\phi)(x) &= (4\pi\varepsilon)^{-n/2}\int_{R^n} e^{-|x-y|^2/4\varepsilon}\phi(y)dy \\ &= (4\pi\varepsilon)^{-n/2}e^{-|x|^2/2\varepsilon}\int_{R^n} e^{|x+y|^2/4\varepsilon}e^{-|y|^2/2\varepsilon}\phi(y)dy \\ &\geqq (4\pi\varepsilon)^{-n/2}e^{-|x|^2/2\varepsilon}\int_{R^n} e^{-|y|^2/2\varepsilon}\phi(y)dy.\end{aligned}$$

This concludes the proof of Theorem 1.

4. Proof of Theorem 2

Recall that $u(t)$ might only satisfy the integral equation (2.1) for almost every $t\in[0,T)$. Now (2.1), Fubini's theorem, and Proposition 1 together imply

$$\|u(t)\|_1=\|\phi\|_1+\int_0^t \|u(s)\|_\gamma^\gamma ds \tag{4.1}$$

for almost every $t\in[0,T)$. Thus, $u(t)\in L^\gamma$ for almost every $t\in[0,T)$; and using Pettis' theorem [10, p. 131] one can show that $u:(0,T)\to L^{\gamma}$ is strongly measurable. Moreover, since $\int_0^t \|u(s)\|_\gamma^\gamma ds<\infty$ for almost all $t\in[0,T)$, it must be true for all $t\in[0,T)$. Thus the right hand side of (2.1) is in $L^1(R^n)$ for all $t\in[0,T)$. It follows that we can modify $u(t)$ for t in some set of measure 0 so that $u(t)$ satisfies the integral equation for all $t\in[0,T)$. In particular, (4.1) holds for all $t\in[0,T)$.

Choose $t_1\in(0,T)$ such that $u(t_1)\in L^{\gamma}$. Then since $u:[t_1,T)\to L^{\gamma}$ is locally essentially bounded, the uniqueness parts of theorems 1 and 4 in [8] guarantee that $u(t)=v(t-t_1)$ for $t\in[t_1,T)$, where $v(t)$ is the continuous solution in L^{γ} to the integral equation (2.1) with initial value $u(t_1)$. (As noted before the statement of Theorem 2, we may use theorem 4 in [8] with $p=\gamma$.) Therefore, we may extend $u(t)$ by letting it equal $v(t-t_1)$, the *maximal* continuous solution in L^{γ} with initial value $u(t_1)$. By (4.1), $u(t)\in L^1$ throughout this extended trajectory.

Replacing ϕ by $u(t_1)$, we may assume from now on that $\phi\in L^1\cap L^{\gamma}$ and that $u:[0,T_\phi)\to L^{\gamma}$ is the maximal solution to (2.1) in L^{γ} with initial value ϕ. T_ϕ is the existence time of the trajectory and we know $\|u(t)\|_\gamma\to\infty$ as $t\to T_\phi$. We will see that $\|u(t)\|_1\to\infty$ as $t\to T_\phi$.

The proof of the existence of local L^{γ} solutions to (2.1) is based on a contraction mapping argument. Since we will need an explicit estimate which comes from this argument, we recall its essential features. (The first two estimates below show why theorem 4 in [8] is valid for $p=\gamma$ if the semigroup is $e^{t\Delta}$.) Let $w:[0,T]\to L^{\gamma}(R^n)$ be a curve with $\|w(t)\|_\gamma\leqq\beta$ for all $t\in[0,T]$. We define

$$\mathcal{F}w(t)=e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}(w(s)^{\gamma})ds.$$

The goal is to prove that $\mathcal{F}$ is a strict contraction on the (non-negative part of the) closed ball of radius β in $L^{\infty}([0,T];L^{\gamma})$. The resulting fixed point of $\mathcal{F}$ will turn out to be a continuous curve in L^{γ}. Using Proposition 1 (d) with $p=1$ and $q=\gamma$, keeping in mind that $n(\gamma-1)/2=1$, we can easily derive the following estimates:

$$\begin{aligned}\|\mathcal{F}w(t)\|_\gamma&\leqq\|\phi\|_\gamma+\int_0^t[4\pi(t-s)]^{-1/\gamma}\|w(s)^{\gamma}\|_1ds\\&\leqq\|\phi\|_\gamma+(4\pi)^{-1/\gamma}\gamma(\gamma-1)^{-1}t^{(\gamma-1)/\gamma}\beta^{\gamma}\end{aligned}$$

and

$$\|\mathcal{F}w_1(t)-\mathcal{F}w_2(t)\|_\gamma\leqq(4\pi)^{-1/\gamma}\gamma(\gamma-1)^{-1}t^{(\gamma-1)/\gamma}\gamma\beta^{\gamma-1}\sup_{[0,t]}\|w_1(s)-w_2(s)\|_\gamma.$$

Thus, there is a constant C such that if $\|\phi\|_\gamma + CT^{(\gamma-1)/\gamma}\beta^\gamma \leqq \beta$, then $\mathcal{F}$ is a strict contraction on the (non-negative part of the) closed ball of radius β in $L^\infty([0,T]; L^\gamma)$. The resulting fixed point of $\mathcal{F}$ will be the first part of the maximal solution to the integral equation (2.1) with the initial value ϕ. Consequently, if $\|\phi\|_\gamma + Ct^{(\gamma-1)/\gamma}\beta^\gamma \leqq \beta$, then $\|u(t)\|_\gamma \leqq \beta$. Applying this to any point in the trajectory, we see that if $0 \leqq s < t$ and

$$(t-s)^{(\gamma-1)/\gamma} \leqq \frac{\beta - \|u(s)\|_\gamma}{C\beta^\gamma},$$

then $\|u(t)\|_\gamma \leqq \beta$.

This estimate enables us to get some control over the asymptotic behavior of u. Observe that if $0 \leqq s < T_\phi$ and $\|u(s)\|_\gamma < \beta$, then

$$(T_\phi - s)^{(\gamma-1)/\gamma} > \frac{\beta - \|u(s)\|_\gamma}{C\beta^\gamma}. \tag{4.2}$$

Indeed, otherwise for some $\beta > \|u(s)\|_\gamma$ and all $t \in (s, T_\phi)$ we would have

$$(t-s)^{(\gamma-1)/\gamma} \leqq \frac{\beta - \|u(s)\|_\gamma}{C\beta^\gamma},$$

which implies $\|u(t)\|_\gamma \leqq \beta$ for all $t \in (s, T_\phi)$ by the previous paragraph. This is impossible since $\|u(t)\|_\gamma \to \infty$ as $t \to T_\phi$. Next, letting $\beta = 2\|u(s)\|_\gamma$ (for example) in (4.2), we see that for $0 < s < T_\phi$

$$(T_\phi - s)^{(\gamma-1)/\gamma} > C'\|u(s)\|_\gamma^{1-\gamma}$$

or

$$\|u(s)\|_\gamma > C''(T_\phi - s)^{-1/\gamma}, \tag{4.3}$$

where C'' is some new fixed constant. This is the desired asymptotic estimate. There are similar estimates for the L^p norms, $\gamma \leqq p < \infty$, and without the restriction that $n(\gamma - 1)/2 = 1$. (See Remark (2) in the last section of this paper.)

The proof of Theorem 2 is completed by substituting (4.3) into (4.1):

$$\|u(t)\|_1 \geqq \|\phi\|_1 + (C'')^\gamma \int_0^t (T_\phi - s)^{-1} ds,$$

and so $\|u(t)\|_1 \to \infty$ as $t \to T_\phi$.

5. Proof of Theorem 3

We first prove part (a). Let

$$C(t)=\left[1-(\gamma-1)\int_0^t \|e^{s\Delta}\phi\|_\infty^{\gamma-1}ds\right]^{-1/(\gamma-1)}$$

Then $C(0)=1$ and $C'(t)=\|e^{t\Delta}\phi\|_\infty^{\gamma-1}C(t)^\gamma$. Consequently

$$C(t)=1+\int_0^t \|e^{s\Delta}\phi\|_\infty^{\gamma-1}C(s)^\gamma ds. \tag{5.1}$$

Now let $u:[0,\infty)\to L^p$ be a continuous curve such that $e^{t\Delta}\phi \leqq u(t)\leqq C(t)e^{t\Delta}\phi$ for all $t\geqq 0$, and let

$$\mathscr{F}u(t)=e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}(u(s)^\gamma)ds.$$

Then

$$\begin{aligned}\mathscr{F}u(t)&\leqq e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}(e^{s\Delta}\phi)^\gamma C(s)^\gamma ds\\ &\leqq e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}(e^{s\Delta}\phi)\|e^{s\Delta}\phi\|_\infty^{\gamma-1}C(s)^\gamma ds\\ &=e^{t\Delta}\phi\left[1+\int_0^t\|e^{s\Delta}\phi\|_\infty^{\gamma-1}C(s)^\gamma ds\right].\end{aligned}$$

Therefore, by (5.1), we have $e^{t\Delta}\phi\leqq \mathscr{F}u(t)\leqq C(t)e^{t\Delta}\phi$ for all $t\geqq 0$.

Now let $u_0(t)=e^{t\Delta}\phi$, $u_1(t)=\mathscr{F}u_0(t)$, and in general $u_{m+1}(t)=\mathscr{F}u_m(t)$. We will show that the $u_m(t)$ converge to the desired solution. Observe first that $u_m(t)\leqq u_{m+1}(t)$ for all $t\geqq 0$. This follows by induction since $u_0(t)\leqq u_1(t)$ for all $t\geqq 0$, and $u(t)\leqq v(t)$ for all $t\geqq 0$ implies $\mathscr{F}u(t)\leqq \mathscr{F}v(t)$ for all $t\geqq 0$. Thus, for each $t\geqq 0$ $u_m(t)$ is a non-decreasing sequence of non-negative functions dominated in $L^p(R^n)$ by $C(t)e^{t\Delta}\phi$. Hence, by the dominated convergence theorem the $u_m(t)$ converge in $L^p(R^n)$ to a function which we call $u(t)$. Clearly, $u(t)\leqq C(t)e^{t\Delta}\phi$ since the same is true of each $u_m(t)$.

Furthermore, the functions $s\mapsto e^{(t-s)\Delta}(u_m(s)^\gamma)$ are dominated by $e^{t\Delta}\phi\|e^{s\Delta}\phi\|_\infty^{\gamma-1}C(s)^\gamma$ in $L^1(0,t;L^p(R^n))$ and converge for each $s\in(0,t)$ to $e^{(t-s)\Delta}(u(s)^\gamma)$ monotonically in $L^p(R^n)$. (The convergence is in L^p for each $s\in(0,t)$ because the dominating function is in L^p for each $s\in(0,t)$.) Consequently, by the dominated convergence theorem for L^p-valued functions, we see that

$$L^p-\lim_{m\to\infty}\int_0^t e^{(t-s)\Delta}(u_m(s)^\gamma)ds=\int_0^t e^{(t-s)\Delta}(u(s)^\gamma)ds.$$

Therefore, if we let $m \to \infty$ in the formula $u_{m+1}(t) = \mathcal{F} u_m(t)$, we get that $u(t) = \mathcal{F} u(t)$; i.e. $u(t)$ is a global solution of (2.1). Continuity of $u(t)$ in L^p easily follows by standard arguments. This proves part (a) of Theorem 3.

We now turn to part (b). We use the fact that local solutions to (2.1) are known to exist for all initial data $\phi \in L^{n(\gamma-1)/2}$. More precisely, choose p such that $1 \leqq p < n(\gamma-1)/2 < p\gamma$. Then theorem 3(b) and corollary 3.1 in [9] guarantee that for every $\phi \geqq 0$ in $L^{n(\gamma-1)/2}$ there exists a non-negative continuous curve $u : [0, T) \to L^{n(\gamma-1)/2}$ which satisfies (2.1) with initial value ϕ. Furthermore, $u(t)$ is continuous into $L^{p\gamma}$ for $t > 0$ and $t^b \| u(t) \|_{p\gamma}$ is bounded near 0, where $b = 1/(\gamma-1) - n/2p\gamma$. In fact, the proof shows that $t^b \| u(t) \|_{p\gamma} \to 0$ as $t \downarrow 0$. (See near the bottom of p. 89 in [9].) Finally, $u(t)$ can be continued as a solution to (2.1) as long as $\| u(t) \|_{p\gamma}$ remains bounded. (The proof in [9] is for a bounded domain Ω, but as mentioned earlier, it is easily modified to include the case $\Omega = R^n$.)

Thus, to show that $u(t)$ can be continued to a global solution, it suffices to show that $\| u(t) \|_{p\gamma}$ can never blow-up. Let $a = n(\gamma-1)/2p\gamma < 1$. Then by Proposition 1 (d) we have that

$$
\begin{aligned}
t^b \| u(t) \|_{p\gamma} &\leqq t^b \| e^{t\Delta} \phi \|_{p\gamma} + t^b \int_0^t \| e^{(t-s)\Delta} (u(s)^\gamma) \|_{p\gamma} ds \\
&\leqq (4\pi)^{-b} \| \phi \|_{n(\gamma-1)/2} + t^b \int_0^t [4\pi(t-s)]^{-a} \| u(s)^\gamma \|_p ds \\
&\leqq (4\pi)^{-b} \| \phi \|_{n(\gamma-1)/2} + (4\pi)^{-a} t^b \int_0^t (t-s)^{-a} s^{-b\gamma} ds \sup_{(0,t)} \| s^b u(s) \|_{p\gamma}^\gamma \\
&= (4\pi)^{-b} \| \phi \|_{n(\gamma-1)/2} + (4\pi)^{-a} \int_0^1 (1-s)^{-a} s^{-b\gamma} ds \sup_{(0,t)} \| s^b u(s) \|_{p\gamma}^\gamma .
\end{aligned}
$$

The last equality follows since $b + 1 - a - b\gamma = 0$. Also, note that $b\gamma < 1$. Therefore, if we let $f(T) = \sup_{(0,T)} \| t^b u(t) \|_{p\gamma}$, we see that $f(T)$ is a continuous, non-decreasing function with $f(0) = 0$ which satisfies

$$
f(T) \leqq (4\pi)^{-b} \| \phi \|_{n(\gamma-1)/2} + C f(T)^\gamma,
$$

where C is a fixed constant. Thus, for sufficiently small $\| \phi \|_{n(\gamma-1)/2}$, $f(T)$ must remain bounded. Indeed, if $C 2^\gamma \alpha^{\gamma-1} < 1$ and $(4\pi)^{-b} \| \phi \|_{n(\gamma-1)/2} \leqq \alpha$, then $f(T)$ can never equal 2α. If it did, we would have $2\alpha \leqq \alpha + C(2\alpha)^\gamma$ or $\alpha \leqq C(2\alpha)^\gamma$, which is false.

This proves that for $\| \phi \|_{n(\gamma-1)/2}$ sufficiently small $t^b \| u(t) \|_{p\gamma}$ must remain bounded, and so proves Theorem 3 part (b).

6. Remarks

(1) The hypothesis in Theorem 1 that the solutions to (2.1) be non-negative is crucial. If $n(\gamma-1)/2 \leqq 1$, there are global classical solutions to (1.1) and (2.1) of mixed sign. Of course, the non-linear term u^{γ} is replaced by $|u|^{\gamma-1}u$ for such solutions. See [4].

(2) The same arguments which led to the estimate (4.3) can be used to show the following. Suppose $\gamma \leqq p < \infty$ and $p > n(\gamma-1)/2$. Let $\phi \in L^p$ and let $u:[0, T_\phi) \rightarrow L^p$ be the maximal solution to (2.1) with initial value ϕ, as shown to exist by theorems 1 and 4 of [8]. If $T_\phi < \infty$, then there is a constant C such that

$$\|u(s)\|_p > C(T_\phi - s)^{n/2p-1/(\gamma-1)}$$

for $0 \leqq s < T_\phi$. This estimate is also correct on a bounded domain in R^n.

(3) The proof of Theorem 3 part (a) is valid for any positivity preserving strongly continuous semigroup on any L^p space.

(4) In order that the hypothesis of Theorem 3 part (a) be met for some $\phi \geqq 0$, it is necessary, because of (3.2), that $n(\gamma-1)/2 > 1$. Thus, there is no conflict with Theorem 1.

(5) If all we know is that $\int_0^\varepsilon \|e^{s\Delta}\phi\|_\infty^{\gamma-1}\,ds < \infty$ for some $\varepsilon > 0$, the proof of Theorem 3 part (a) shows that there exists a local solution to (2.1) which exists as long as $(\gamma-1)\int_0^t \|e^{s\Delta}\phi\|_\infty^{\gamma-1}\,ds < 1$. This solution satisfies (2.2) for all such t.

(6) There is an interesting relationship between the hypotheses of parts (a) and (b) of Theorem 3. Suppose $n(\gamma-1)/2 > 1$ and $n = 1$ or 2. If $\|\phi\|_{n(\gamma-1)/2}$ is sufficiently small, then $(\gamma-1)\int_0^\infty \|e^{t\Delta}\phi\|_\infty^{\gamma-1}\,dt \leqq 1$. We prove this using the Marcinkiewicz interpolation theorem. (See [7] p. 272.) Consider the map H which takes a function ϕ in $L^p(R^n)$ into the curve $u(t) = \|e^{t\Delta}\phi\|_\infty$ on $(0, \infty)$. The map H is clearly subadditive, and by Proposition 1 (d) it is of weak-type (p, s) whenever $1 \leqq p \leqq \infty$ and $s = 2p/n$. (Since $n \leqq 2$, we always have $p \leqq s$, which is necessary in order to apply the interpolation theorem.) Thus, by the Marcinkiewicz theorem, H is of strong-type (p, s) whenever $1 < p < \infty$ and $s = 2p/n$, i.e.

$$\left(\int_0^\infty \|e^{t\Delta}\phi\|_\infty^s\,dt\right)^{1/s} \leqq C\|\phi\|_p$$

for some constant C independent of ϕ. Letting $p = n(\gamma-1)/2$, we get the desired result.

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